Matrix Lie Groups and Lie Algebras

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Abstract This reading project contains very basic Lie groups, Lie algebras and representation theory. I learned this stuff after the course MATH 5070 because I want to figure out the theory of Lie algebras and Lie groups more clearly. This will contains an introduction to matrix Lie groups, Lie algebras, the matrix exponential, as well as some representation theory and the Baker–Campbell–Hausdorff formula. The context is not hard and I try to explain my understanding toward this stuff in my writing.

1 Introduction

Lie group is an important object in mathematics and physics. A Lie group is a group of symmetries where the symmetries varies smoothly. More precisely, a Lie group admits : a group structure, a topological manifold structure, and a smooth structure. Moreover, these structures are compatible, which makes it into important objects as well as tools in math.
In this notes, I will briefly introduce contents below:

1. Matrix Lie groups, Lie algebras, and their relations
2. Baker–Campbell–Hausdorff formula
3. Representation of $sl(2, \mathbb{C})$

1.1 Lie groups: definitions and examples

What is a Matrix Lie group?

Definition 1 (Matrix Lie group) A matrix Lie group is a closed subset in $GL(n; \mathbb{C})$.

Remark 1 Here we only require closedness in $GL(n;\mathbb{C})$, this means that a matrix Lie group does't need to be a closed subset of $M(n;\mathbb{C})$.

Example 1 **The special linear group** $SL(n)$
The special linear group (over **R** or **C**) is the group of $n\times n$ invertible matrices having determinant one.

Example 2 **One dimensional unitary group** $U(1)$
For a complex variable z, consider the phase transformation $z \to z^{'}= e^{i\theta}z$, where θ is a real constant parameter. We can write this as $U(\theta)z = e^{i\theta}z$. Identifying the Hermitian conjugate $U^{\dagger}(\theta) = U^{-1}(\theta) = U(-\theta)$, so that the operator $U(\theta)$ is unitary, $U(\theta)U^{\dagger}(\theta) = 1 = U^{\dagger}(\theta)\mathrm{U}(\theta)$, the set of phase transformations $U(1) = \{U(\theta), 0 \leq \theta < 2\pi\}$ forms $U(1)$, the one dimensional unitary group.

Example 3 **unitary group** $U(n)$
Generalize the group of one dimensional phase transformations gives us $U(n)$, where the element a_{ij} of a matrix A in $U(n)$ are complex parameters such that $A^{\dagger}A = AA^{\dagger} = I_n$.

Example 4 **Special unitary group** $SU(n)$
We consider matrices in $U(n)$ with determination 1, which is a closed subset of $M(n;\mathbb{C})$, hence a matrix Lie group.

Another way of looking at $U(n)$ is through introducing the standard inner product on $\mathbf{C^n}$. Here we put the conjugate on the first factor when we talks about the inner product: $\langle x, y\rangle = \sum \bar{x_j} y_j$. Say matrix A is unitary if it satisfies: $\langle x, y\rangle = \langle Ax, Ay\rangle$, which, in other words, is inner product preserving.

Consider now the bilinear form $\mathbf{C^n}$ defined by: $(x, y) = \sum x_j y_j$. Say matrix A is **the complex orthogonal group** $O(n;\mathbb{C})$ if it satisfies: $(x, y) = (Ax, Ay)$. Those matrices with determination 1 in $O(n;\mathbb{C})$ are called **the special orthogonal group** $SO(n;\mathbb{C})$.

Further generalize it, now consider the bilinear form on $\mathbf{C^n}$ defined by: $[x, y]_{n,k} = x_1y_1 + x_2y_2 + \cdots + x_ny_n - x_{n+1}y_{n+1} - x_{n+2}y_{n+2} - x_{n+k}y_{n+k}$. Say a matrix A is **the generalized orthogonal group** $O(n; k)$ if it satisfies: $[x, y]_{n,k} = [Ax, Ay]_{n,k}$. Those matrices with determination 1 in $O(n; k)$ are denoted as $SO(n; k)$.

Example 5 **The Heisenberg Group**
The set of all 3×3 real matrices A of the form below, is the Heisenberg group.
A= $\begin{pmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & c \end{pmatrix}$, where a, b, c are real numbers.

Definition 2 **((General) Lie group)** A Lie group is a smooth manifold equipped with a group structure such that the operations of group multiplication $m : G \times G \to G$ by $m(g_1, g_2) = g_1g_2$ and inversion $i : G \to G$ by $i(g) = g^{-1}$ are smooth.

Why we require the smoothness of group multiplication/inversion here? We need to ensure the compatibility of the manifold structure and the group structure of a Lie group.

Matrix Lie groups are all Lie groups since every matrix Lie group G can be viewed as a smooth embedded submanifold of $M(n; \mathbf{C})$.

Definition 3 (Lie group morphisms) Let G and H be matrix Lie groups. A map Φ from G to H is called a Lie group homomorphism, if

1. Φ is a group homomorphism
2. Φ is continuous.

1.2 Lie Algebras: definitions and examples

We now introduce the notion of a Lie algebra. After that, we associate to each matrix Lie group a Lie algebra.

Definition 4 (Finite-dimensional Lie algebra)
A finite dimensional Lie algebra is a real/complex vector space g, together with a map [,] from $g \times g$ into g, satisfying the following properties:

1. bilinearity: $[ax + by, c] = a[x, c] + b[y, c]$, $[x, by + cz] = b[x, y] + c[x, z]$.
2. skew symmetric: $[x, y] = -[y, x]$.
3. the Jacobi Identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

Say two elements x, y in Lie algebra commute if $[x, y] = 0$.The map [,] is referred to as the bracket operation on g.

Remark 2 Due to our definition, a Lie algebra doesn't need to be an algebra, in contrast to what it suggests in its name. However, we can always define a Lie algebra structure on an algebra.

Example 6 **Lie algebra on an associative algebra**
Let $\mathcal{A}$ be an associative algebra and g is a subspace of $\mathcal{A}$ which is closed under the bracket operation: $[X, Y] = XY - YX$, for any X, Y in g. And g forms a Lie algebra. Note: In this example, to check the Jacobi Identity of the Lie algebra, the associativity of g is important.

Remark 3 The Jacobi identity seems to say, the bracket operation behaves as if it were $XY - YX$ in an associative algebra, even explicitly it may be defined in an distinct way.

Example 7 $sl(n; \mathbf{C})$ $sl(n; \mathbf{C})$ consists of matrices with trace 0. It is a Lie algebra under the bracket: $[X, Y] = XY - YX$.

Definition 5 (subalgebra of Lie algebra) A subalgebra h of Lie algebra g is a subspace, which is closed under brackets. It is said to be an ideal if $[X, H] \in h$, for all $X \in g$ and $H \in h$.

Definition 6 (Lie algebra morphisms) Let g, h be two Lie algebras. A linear map $\phi : g \to h$ is called a Lie algebra homomorphism if it preserves the bracket operation. That is, $\phi[X, Y] = [\phi(X), \phi(Y)]$ for all X, Y in g.

Definition 7 (Adjoint map) g is a Lie algebra and X is an element of g, we can define a linear map from g to itself using the bracket operation. That is,
$ad_x(Y) = [X, Y]$.

Using this useful concept, we rewrite our Jacobi identity as $ad_x([Y, Z]) = [ad_x(Y), Z] + [Y, ad_x(Z)]$, which displays the derivation property of this adjoint map.

Proposition 1 *ad is a Lie algebra homomorphism from g to End(g). In other words,* $ad_{[X,Y]} = [ad_X, ad_Y]$.

1.3 From Lie groups to Lie algebras

We now have Lie groups, Lie algebras, Lie groups homomorphisms, Lie algebra homomorphisms, etc.. But what's the relation among them? We next assign each Lie group with a Lie algebra. To do that, we firstly review some matrix theory.

1.3.1 The Matrix Exponential

If X is an $n \times n$ matrix, we define the exponential of X, denoted as e^X or $expX$, using the Taylor series:

$$e^X = \sum_{i=1}^{n} \frac{X^m}{m!} \tag{1}$$

It can be shown that the series (1) converges for every matrices in $M(n; \mathbf{C})$. Furthermore, e^X is a continuous function with respect to X.

We list some properties/theorems of the matrix exponential.

Proposition 2 *1. e^X is invertible and its inverse is $e^{(-X)}$.*
2. $e^{(\alpha+\beta)X} = e^{(\alpha X)}e^{(\beta X)}$.
3. If XY = YX, then $e^{X+Y} = e^X e^Y = e^Y e^X$.
4. If $C \in GL(n; \mathbf{C})$, then $e^{CXC^{-1}} = Ce^X C^{-1}$.
5. $det(e^x) = e^{trace(X)}$.

Theorem 1 *Let X be a $n \times n$ complex matrix. Then $e^t X$ is a smooth curve in $M(n; \mathbf{C})$ and $\frac{dy}{dt} e^{tX} = Xe^{tX} = e^{tX} X$. We observe that $\frac{dy}{dt}|_{t=0} = X$.*

Theorem 2 *The exponential map is an infinitely differentiable map of $M(n;\mathbf{C})$ into $M(n;\mathbf{C})$.*

Definition 8 (One-parameter subgroup) A function A :$\mathbf{R} \rightarrow GL(n;\mathbf{C})$ is called a one-parameter subgroup of $GL(n;\mathbf{C})$ if :

1. A is continuous.
2. $A(0) = I$.
3. $A(t+s) = A(t)A(s)$.

Theorem 3 *If $A(\cdot)$ is a one-parameter subgroup of $GL(n;\mathbf{C})$, then there exists a unique $n \times n$ complex matrix X such that: $A(t) = e^{tX}$.*

1.3.2 The Lie algebra of a Lie group

Given a matrix Lie group G, we define the corresponding Lie algebra g as: $g = \{X : e^{tX} \in G, \text{for any } t \in R\}$.

We now establish useful properties of the Lie algebra induced by a Lie group.

Having such a defined "Lie algebra" induced from a Lie group in hand, we ask ourselves, which operation is allowed?

Proposition 3 *Let G be a matrix Lie group with Lie algebra g, $X, Y \in g$ then we have:*

1. *$AXA^{-1} \in g$, for all $\in g$.*
2. *$sX \in g$ for all real numbers s.*
3. *$X + Y \in g$.*
4. *$XY - YX \in g$.*

Thus, by introducing the common bracket $[X, Y] = XY - YX$, we get the Lie algebra of a matrix Lie group as a real Lie algebra.

Example 8 **The Lie algebra of $GL(n;\mathbf{C})$**
$gl(n,\mathbf{C})$ (which refers to the corresponding matrix of $GL(n;\mathbf{C})$), is the space of all $n \times n$ matrices with complex entries.

Example 9 **The Lie algebra of $SL(n;\mathbf{C})$**
$sl(n,\mathbf{C})$ (which refers to the corresponding matrix of $SL(n;\mathbf{C})$), is the space of all $n \times n$ matrices with complex entries with trace 0.

Example 10 **The Lie algebra of $U(n;\mathbf{C})$**
$u(n,\mathbf{C})$ (which refers to the corresponding matrix of $U(n;\mathbf{C})$), is the space of all $n \times n$ matrices with complex entries satisfying $A^{\star} = -A$, where $A^{\star}$ is the conjugate of the matrix.

Example 11 **The Lie algebra of $SU(n;\mathbf{C})$**
$su(n,\mathbf{C})$ (which refers to the corresponding matrix of $SU(n;\mathbf{C})$), is the space of all $n \times n$ matrices with complex entries satisfying $A^{\star} = -A$ with trace 0 ($A^{\star}$ is the conjugate of the matrix) .

2 Relations: Lie algebras and Lie groups

For this reading notes, we try to use Lie algebras to study the structure of a Lie group. In this section, we try to find out some similarities and correspondence between Lie algebras and Lie groups, such as the correspondence of morphisms and substructures.

2.1 Relations between commutativity

Theorem 4 *If G is commutative then g is also commutative.*

Proof For any two matrices $X;Y \in M(n,\mathbf{C})$, the commutator of X and Y can be computed through:

$$[X,Y] = \frac{d}{dt}(\frac{d}{ds}e^{tX}e^{sY}e^{-tX}|_{s=0})|_{t=0} \tag{2}$$

If G is commutative and X and Y $\in G$, we know that e^{tX} commutes with e^{sY},m which means the right hand side is independent of t, thus $[X,Y] = 0$. □

2.2 Relations between morphisms

The following theorem tells us that a Lie group homomorphism between two Lie groups gives rise to a map between the corresponding Lie algebras.

Theorem 5 *Let G and H be matrix Lie groups, with Lie algebras g, h. When we have a Lie group homomorphism $\Phi : G \to H$, there is a unique real-linear map $\phi : g \to h$ such that:*

$$\Phi(e^X) = e^{\phi(X)} \tag{3}$$

for all $X \in g$. The map ϕ has following additional properties:

1. $\phi(AXA^{-1}) = \Phi(A)\phi(X)\Phi(A)^{-1}$, for all $X \in g$, $A \in G$.
2. $\phi([X,Y]) = (\phi[X],\phi[Y])$ for all $X,Y \in g$.
3. $\phi(X) = \frac{d}{dt}\Phi(e^{tX})|_{t=0}$.

Proof Since $\Phi : G \to H$ is a Lie group homomorphism, then for each X$\in g$, $\Phi(e^{tX})$ will be a a one-parameter subgroup of H. Thus there will be an unique matrix Z such that:

$$\Phi(e^{tX}) = e^{tZ} \tag{4}$$

for all t $\in \mathbf{R}$. We define $\phi(X) = Z$, and check ϕ satisfies required properties. Taking t=1, we see that $\Phi(e^X) = e^{\phi(X)}$ for all X in g. Moreover, since $\Phi(e^{tX}) = e^{tZ}$ for all t, then $\Phi(e^{tsX}) = e^{tsZ}$, showing that $\phi(sX) = s\phi(X)$. Using the formula:

$$e^{X+Y} = \lim_{n\to\infty}(e^{\frac{X}{m}}e^{\frac{Y}{m}})^m \tag{5}$$

as well as the continuity of Φ, we get:

$$e^{t\phi(X+Y)} = \Phi(\lim_{m\to\infty}(e^{\frac{tX}{m}}e^{\frac{tY}{m}})^m) = \lim_{m\to\infty}(\Phi(e^{\frac{tX}{m}})\Phi(e^{\frac{tY}{m}}))^m \tag{6}$$

Thus,

$$e^{t\phi(X+Y)} = \lim_{m\to\infty}((e^{\frac{t\phi(X)}{m}}e^{\frac{t\phi(Y)}{m}}))^m = e^{t(\phi(X)+\phi(Y))} \tag{7}$$

Differentiating this result at $t = 0$, we get $\phi(X+Y) = \phi(X) + \phi(Y)$.

Thus, We have thus obtained a real-linear map ϕ satisfying $\Phi(e^X) = e^{\phi(X)}$. If this also hold true for another linear map ϕ' with this property, we would have:

$$e^{t\phi(X)} = e^{t\phi(X)'} = \Phi(e^{tX}) \tag{8}$$

for all $t \in \mathbf{R}$, by differentiating this result at $t = 0$, we get $\phi(X) = \phi'(X)$.

We now verify the remaining claimed properties of ϕ. For any $A \in G$, we have

$$e^{t\phi(AXA^{-1})} = e^{\phi(tAXA^{-1})} = \Phi(e^{tAXA^{-1}}) \tag{9}$$

Thus,

$$e^{t\phi(AXA^{-1})} = \Phi(A)\Phi(e^{tX})\Phi(A)^{-1} = \Phi(A)e^{t\phi(X)}\Phi(A)^{-1}. \tag{10}$$

Differentiating this identity at t=0 gives property 1. What's more, for all $X, Y \in g$, we have

$$\phi([X,Y]) = \phi(\frac{d}{dx}e^{tX}Ye^{-tX}|_{t=0}) = \frac{d}{dt}\phi(e^{tX}Ye^{-tX})|_{t=0}, \tag{11}$$

using the fact that a derivative commutes with linear transformation. Thus,

$$\begin{aligned}\phi([X,Y]) &= \frac{d}{dt}\Phi(e^{tX})\phi(Y)\phi(e^{-tX})|_{t=0} \\ &= \frac{d}{dt}e^{t\phi(X)}\phi(Y)e^{-t\phi(X)}|_{t=0} = (\phi[X],\phi[Y])\end{aligned} \tag{12}$$

which gives 2. Thirdly, since $\Phi(e^{tX}) = e^{\phi(tX)} = e^{t\phi(X)}$, we can compute $\phi(X)$ as in point 3. □

Remark 4 From here, we konw that every Lie group homomorphism gives rise to a Lie algebra homomorphism. Will the reverse version be true? We will solve this question in the next section.

2.3 Relations between substructures

Given a subgroup in a Lie group, could we obtain a subalgebra in the corresponding Lie algebra?

Theorem 6 *Let G and H be matrix Lie groups with $H \subset G$, the Lie algebra h of H is a subalgebra of the Lie algebra g of G. Furthormore,*

1. *H is a normal subgroup of G, then h is an ideal in g.*
2. *If G and H are connected and h is an ideal in g, then H is a normal subgroup in G.*

3 Converse Version: From Lie algebras to Lie groups

We can construct objects/find morphisms of a Lie group using Lie algebra. Just as we know,

1. Every matrix Lie group has its own finite dimensional Lie algebra.
2. Let G and H be matrix Lie groups, with Lie algebras g, h. When we have a Lie group homomorphism $\Phi : G \to H$, there is a unique real-linear map $\phi : g \to h$ such that:
$$\Phi(e^X) = e^{\phi(X)} \tag{13}$$
for all $X \in g$ with several properties.
3. A subgroup H of a matrix Lie group G corresponds to a subalgebra h of the Lie algebra g.

However, it's often not enough to consider just one direction. So we ask ourselves: under which assumptions can I produce the reversed direction? We will give a quick answer, whose proof relies on the Baker–Campbell–Hausdorff formula.

1. True.
2. True, given that G is simply connected.
3. True, if H is a connected Lie group.

We skip details and simply list the BCH formula/several statements below.

3.1 The Baker–Campbell–Hausdorff Formula

This proof of the converse version will relies on the Baker–Campbell–Hausdorff formula, stated as follows:

Theorem 7 *For all $n \times n$ complex matrices X with the norm of X, Y sufficiently small, we have:*

$$log(e^X e^Y) = X + \int_0^1 g(e^{ad_X})(e^{tad_Y})(Y)dt \tag{14}$$

Remark 5 Not only do we focus on the Baker–Campbell–Hausdorff (BCH) itself, but we notice that this conveys the message: $log(e^X e^Y)$ can be represented in terms of communtators involving only X and Y. Besides, the formula implies that all information about the product operation on a matrix Lie group, at least near the identity, is encoded in the Lie algebra.

Using BCH formula (mainly by the constructions local homomorphism from ϕ), we can prove the following:

Theorem 8 (Answer to reversed Q2) *Let G and H be matrix Lie groups with Lie algebras g and h respectively, and let $\phi : g \to h$ be a a Lie algebra homomorphism. If G is simply conncected, there exists a unique Lie group homomorphism $\Phi : G \to H$, such that*

$$\Phi(e^X) = e^{\phi(X)} \tag{15}$$

for all $X \in g$.

Theorem 9 (Answer to reversed Q3) *Let H be a connected Lie subgroup of $GL(n; \mathbf{C})$ with Lie algebra h. Then H can be given the structure of a smooth manifold in such a way that the group operations on H are smooth and the inclusion map of H into $GL(n; \mathbf{C})$ is smooth.*

Thus, every connected matrix Lie subgroup can be made into a Lie group.

3.2 Lie's Third Theorem

Theorem 10 (Answer to reversed Q1) *If g is any finite-dimensional, real Lie algebra, there exists a connected Lie subgroup G of $GL(n; \mathbf{C})$ whose Lie algebra is isomorphic to g.*

Theorem 11 *Every finite-dimensional, real Lie algebra is isomorphic to the Lie algebra of some matrix Lie group.*

4 Representations of $sl(2; \mathbf{C})$

4.1 Some Representation theory

Measuring an algebraic structure is difficult. We sometimes complete this by letting the algebraic structure acts on a vector space, and character algebraic structures by the actions they behave on other vector spaces. We next consider the representations of Lie groups and Lie algebras simultaneously. Firstly let me give the definition.

Definition 9 Let G be a matrix Lie group. A finite-dimensional complex/real representation of G is a Lie group homomorphism : $\Pi : G \to GL(V)$, here V is a finite-dimensional complex/real vector space.
Let g be a complex/real Lie algebra, a finite-dimensional complex representation of g is a Lie algebra homomorphism $\pi : g \to gl(V)$, here V is a finite-dimensional complex/real vector space.

Theorem 12 *Let G be a matrix Lie group with Lie algebra g and let* Π *be a finite-dimensional representation of G acting on V, Then there is a unique representation* π *of g acting on the same space such that :*

$$\Pi(e^X) = e^{\pi(X)} \tag{16}$$

for all $X \in g$*. The representation* π *can be computed as* $\pi(X) = \frac{d}{dt}\Pi(e^{tX})|_{t=0}$.

Combining the representations of Lie groups and Lie algebras, we ask ourselves, whether every representation π of g comes from a representation Π of G. (This is true if G is simply connected).

Some representations can be divided into simpler representations, and under some representations different elements will induce different elements in $GL(V)$ (Linear Transformation over V). We list several kinds of representations below:

1. If this homomorphism is injective, then call the representation a faithful representation.
2. A representation with no nontrivial invariant subspaces is called irreducible.

We can see some examples of representations of Lie groups.

1. The easiest one : trivial representation $\Pi : G \to GL(1;\mathbf{C})$, by sending every element g ∈ G to the identity element of $GL(1;\mathbf{C})$.
2. The adjoint representation: $Ad : G \to GL(g)$ given by $A \mapsto Ad_A$.

We can also write the corresponding version for Lie algebras.

1. trivial representation $\pi : g \to gl(1;\mathbf{C})$, by $\pi(X) = 0$, for all $X \in g$.
2. The adjoint representation: $ad : g \to gl(g)$ given by $X \mapsto ad_X$.

Given a representation, when we want to restrict it, we introduce the invariant subspace. When we want to extend it, we consider the direct sum of representations. We omit the definitions here.

We can further see the relations between Lie groups and Lie algebras as follows:

Theorem 13 *Let G be a connected matrix Lie group with Lie algebra* g*. Let* Π *be a representation of G and* π *be the associated representation of* g*.* Π *is irreducible if and only if* π *is irreducible.*

Theorem 14 *Let G be a connected matrix Lie group with Lie algebra* g*. Let* Π_1, Π_2 *be representations of G and* π_1, π_2 *be the associated representations of* g*. Then* Π_1, Π_2 *are isomorphic if and only if* π_1, π_2 *are isomorphic.*

4.2 Schur's lemma

Having a Lie group G, we can use it to act on different vector spaces V, W. And denote the representations as Π, Σ. Say a linear map $\phi : V \to W$ an intertwining map of representations if:

$$\phi(\Pi(A)v) = \Sigma(A)\phi(V) \tag{17}$$

The analogous property defines intertwining maps of representations of a Lie algebra. Additionally, if ϕ is invertible, then ϕ is said to be an isomorphism of representations. We now introduce Schur's lemma, which reveals orthogonality relations.

Theorem 15 (Schur's lemma) *Let V, W be irreducible representations of G.*

1. *If f : V → W is a G-morphism, then either f 0, or f is invertible.*
2. *If $f_1, f_2 : V \to W$ are two G-morphisms and $f_2 \neq 0$, then there exists $\lambda \in \boldsymbol{C}$ such that $f_1 = \lambda f_2$.*

Proof (1) Suppose f is not identically zero. Since $ker(f)$ is a G-invariant subset in V, it must be 0. So f is injective. In particular, $f(V)$ is a nonzero subspace of W. On the other hand, we can check that $f(V)$ is a G-invariant subspace of W. It follows that $f(V) = W$, and thus f is invertible.
(2) Since $f_2 \neq 0$, it is invertible. is a G-morpism from V to V itself. Let λ be one of the eigenvalues of the linear map f. Then $f - \lambda I$ is a G-morphism from V to V which is not invertible. Thus, $f - \lambda I$ =0, $f_1 = \lambda f_2$. □

4.3 A specific example: $sl(2; \mathbf{C})$

In this subsection, we will show (up to isomorphism) all of the finite-dimensional irreducible complex representations of the Lie algebra $sl(2; \mathbf{C})$. Firstly recall that $sl(2; \mathbf{C})$ consists of matrices with trace 0. We use the following basis for $sl(2; \mathbf{C})$:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{18}$$

We just mention the theorem, but the details in the proof is worthwhile to learn.

Theorem 16 *If (π, V) is a finite-dimensional representation of $sl(2; \mathbf{C})$, the following results hold.*

1. *Every eigenvalue of $\pi(H)$ is an integer. Furthermore, if v is an eigenvector for $\pi(H)$ with eigenvalue λ and $\pi(X)v = 0$, then λ is a non-negative integer.*
2. *The operators $\pi(X)$ and $\pi(Y)$ are nilpotent.*
3. *If we define $S : V \to V$ by*

$$S = e^{\pi(X)} e^{-\pi(Y)} e^{\pi(X)}, \tag{19}$$

then S *satisfies* $S\pi(H)S^{-1} = -\pi(H)$.
4. *If an integer* k *is an eigenvalue for* $\pi(H)$*, so is each of the numbers* $-|K|, -|K|+2, \cdots, |K|-2, |K|$.

References

Hall, Brian C.. "Lie Groups, Lie Algebras, and Representations: An Elementary Introduction." (2004).